

1. The differential displacements in the Cartesian, cylindrical, and spherical systems are, respectively,

$$d\mathbf{l} = dx \mathbf{a}_x + dy \mathbf{a}_y + dz \mathbf{a}_z$$

$$d\mathbf{l} = d\rho \mathbf{a}_\rho + \rho d\phi \mathbf{a}_\phi + dz \mathbf{a}_z$$

$$d\mathbf{l} = dr \mathbf{a}_r + r d\theta \mathbf{a}_\theta + r \sin \theta d\phi \mathbf{a}_\phi$$

Note that  $d\mathbf{l}$  is always taken to be in the positive direction; the direction of the displacement is taken care of by the limits of integration.

2. The differential normal areas in the three systems are, respectively,

$$dS = dy dz \mathbf{a}_x$$

$$dx dz \mathbf{a}_y$$

$$dx dy \mathbf{a}_z$$

$$dS = \rho d\phi dz \mathbf{a}_\rho$$

$$d\rho dz \mathbf{a}_\phi$$

$$\rho d\rho d\phi \mathbf{a}_z$$

$$dS = r^2 \sin \theta d\theta d\phi \mathbf{a}_r$$

$$r \sin \theta dr d\phi \mathbf{a}_\theta$$

$$r dr d\theta \mathbf{a}_\phi$$

Note that  $dS$  can be in the positive or negative direction depending on the surface under consideration.

3. The differential volumes in the three systems are

$$dv = dx dy dz$$

$$dv = \rho d\rho d\phi dz$$

$$dv = r^2 \sin \theta dr d\theta d\phi$$

4. The line integral of vector  $\mathbf{A}$  along a path  $L$  is given by  $\int_L \mathbf{A} \cdot d\mathbf{l}$ . If the path is closed, the line integral becomes the circulation of  $\mathbf{A}$  around  $L$ , that is,  $\oint_L \mathbf{A} \cdot d\mathbf{l}$ .
5. The flux or surface integral of a vector  $\mathbf{A}$  across a surface  $S$  is defined as  $\int_S \mathbf{A} \cdot d\mathbf{S}$ . When the surface  $S$  is closed, the surface integral becomes the net outward flux of  $\mathbf{A}$  across  $S$ , that is,  $\oint_S \mathbf{A} \cdot d\mathbf{S}$ .
6. The volume integral of a scalar  $\rho_v$  over a volume  $v$  is defined as  $\int_v \rho_v dv$ .
7. Vector differentiation is performed by using the vector differential operator  $\nabla$ . The gradient of a scalar field  $V$  is denoted by  $\nabla V$ , the divergence of a vector field  $\mathbf{A}$  by  $\nabla \cdot \mathbf{A}$ , the curl of  $\mathbf{A}$  by  $\nabla \times \mathbf{A}$ , and the Laplacian of  $V$  by  $\nabla^2 V$ .
8. The divergence theorem,  $\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} dv$ , relates a surface integral over a closed surface to a volume integral.
9. Stokes's theorem,  $\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$ , relates a line integral over a closed path to a surface integral.
10. If Laplace's equation,  $\nabla^2 V = 0$ , is satisfied by a scalar field  $V$  in a given region,  $V$  is said to be harmonic in that region.
11. A vector field is solenoidal if  $\nabla \cdot \mathbf{A} = 0$ ; it is irrotational or conservative if  $\nabla \times \mathbf{A} = \mathbf{0}$ .
12. A summary of the vector calculus operations in the three coordinate systems is provided on the inside back cover of the text.
13. The vector identities  $\nabla \cdot \nabla \times \mathbf{A} = 0$  and  $\nabla \times \nabla V = \mathbf{0}$  are very useful in EM. Other vector identities are in Appendix A.10.

### REVIEW QUESTIONS

3.1 Consider the differential volume of Figure 3.25. Match the items in the left-hand column with those on the right.

- |                                 |                            |
|---------------------------------|----------------------------|
| (a) $d\mathbf{l}$ from A to B   | (i) $dy dz \mathbf{a}_x$   |
| (b) $d\mathbf{l}$ from A to D   | (ii) $-dx dz \mathbf{a}_y$ |
| (c) $d\mathbf{l}$ from A to E   | (iii) $dx dy \mathbf{a}_z$ |
| (d) $d\mathbf{S}$ for face ABCD | (iv) $-dx dy \mathbf{a}_z$ |
| (e) $d\mathbf{S}$ for face AEHD | (v) $dx \mathbf{a}_x$      |
| (f) $d\mathbf{S}$ for face DCGH | (vi) $dy \mathbf{a}_y$     |
| (g) $d\mathbf{S}$ for face ABFE | (vii) $dz \mathbf{a}_z$    |

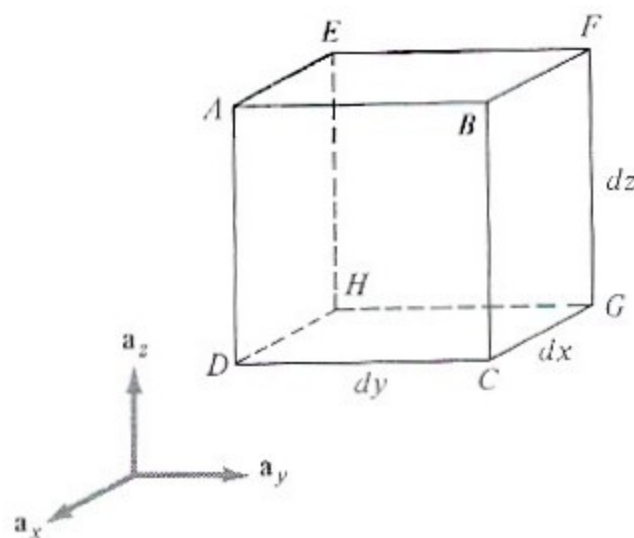


FIGURE 3.25 For Review Question 3.1.

3.2 For the differential volume in Figure 3.26, match the items in the left-hand list with those on the right.

- |                                   |                                        |
|-----------------------------------|----------------------------------------|
| (a) $d\mathbf{l}$ from $E$ to $A$ | (i) $-\rho d\phi dz \mathbf{a}_\rho$   |
| (b) $d\mathbf{l}$ from $B$ to $A$ | (ii) $-d\rho dz \mathbf{a}_\phi$       |
| (c) $d\mathbf{l}$ from $D$ to $A$ | (iii) $-\rho d\rho d\phi \mathbf{a}_z$ |
| (d) $dS$ for face $ABCD$          | (iv) $\rho d\rho d\phi \mathbf{a}_z$   |
| (e) $dS$ for face $AEHD$          | (v) $d\rho \mathbf{a}_\rho$            |
| (f) $dS$ for face $ABFE$          | (vi) $\rho d\phi \mathbf{a}_\phi$      |
| (g) $dS$ for face $DCGH$          | (vii) $dz \mathbf{a}_z$                |

3.3 Consider the object shown in Figure 3.8. For the volume element, match the items in the left-hand column with those on the right.

- |                                   |                                                   |
|-----------------------------------|---------------------------------------------------|
| (a) $d\mathbf{l}$ from $A$ to $D$ | (i) $-r^2 \sin \theta d\theta d\phi \mathbf{a}_r$ |
| (b) $d\mathbf{l}$ from $E$ to $A$ | (ii) $-r \sin \theta dr d\phi \mathbf{a}_\theta$  |
| (c) $d\mathbf{l}$ from $A$ to $B$ | (iii) $r dr d\theta \mathbf{a}_\phi$              |
| (d) $dS$ for face $EFGH$          | (iv) $dr \mathbf{a}_r$                            |
| (e) $dS$ for face $AEHD$          | (v) $r d\theta \mathbf{a}_\theta$                 |
| (f) $dS$ for face $ABFE$          | (vi) $r \sin \theta d\phi \mathbf{a}_\phi$        |

3.4 If  $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ , the position vector of point  $(x, y, z)$  and  $r = |\mathbf{r}|$ , which of the following is incorrect?

- |                                   |                                                 |
|-----------------------------------|-------------------------------------------------|
| (a) $\nabla r = \mathbf{r}/r$     | (c) $\nabla^2(\mathbf{r} \cdot \mathbf{r}) = 6$ |
| (b) $\nabla \cdot \mathbf{r} = 1$ | (d) $\nabla \times \mathbf{r} = \mathbf{0}$     |

3.5 Which of the following is a mathematically incorrect expression?

- |              |               |
|--------------|---------------|
| (a) grad div | (c) grad curl |
| (b) div curl | (d) curl grad |

3.6 Which of the following is zero?

- |              |               |
|--------------|---------------|
| (a) grad div | (c) curl grad |
| (b) div grad | (d) curl curl |

3.7 Given field  $\mathbf{A} = 3x^2yza_x + x^3za_y + (x^3y - 2z)\mathbf{a}_z$ , it can be said that  $\mathbf{A}$  is

- |                    |                  |
|--------------------|------------------|
| (a) Harmonic       | (d) Rotational   |
| (b) Divergenceless | (e) Conservative |
| (c) Solenoidal     |                  |

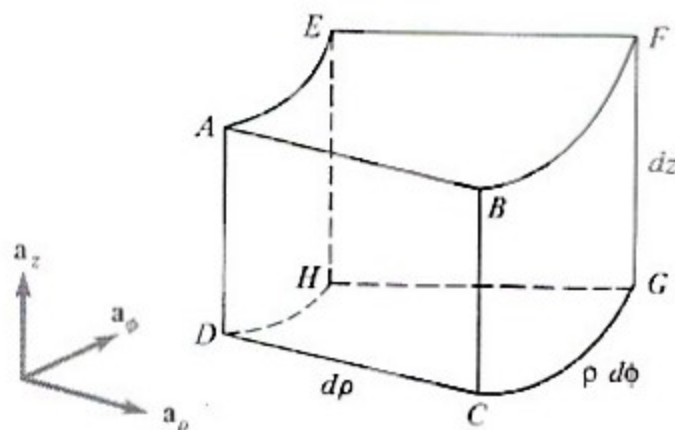


FIGURE 3.26 For Review Question 3.2.



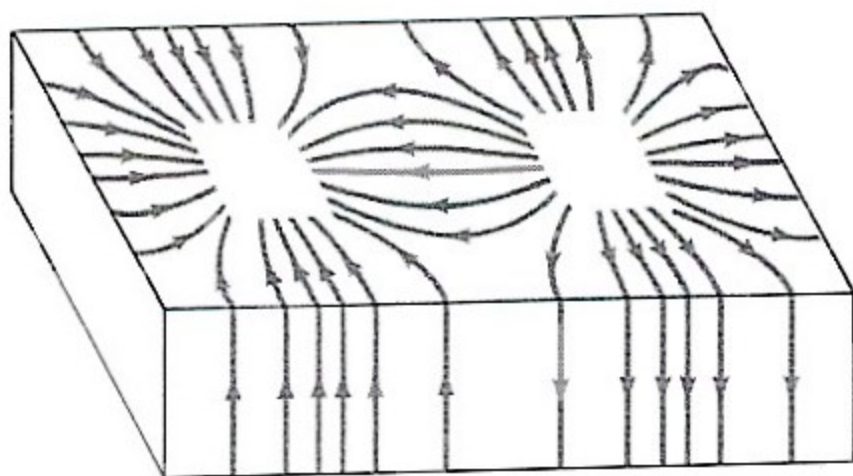


FIGURE 3.27 For Review Question 3.8.

3.8 The surface current density  $\mathbf{J}$  in a rectangular waveguide is plotted in Figure 3.27. It is evident from the figure that  $\mathbf{J}$  diverges at the top wall of the guide, whereas it is divergenceless at the side wall.

(a) True

(b) False

3.9 Stokes's theorem is applicable only when a closed path exists and the vector field and its derivatives are continuous within the path.

(a) True

(c) Not necessarily

(b) False

3.10 If a vector field  $\mathbf{Q}$  is solenoidal, which of these is true?

(a)  $\oint_L \mathbf{Q} \cdot d\mathbf{l} = 0$ (d)  $\nabla \times \mathbf{Q} \neq 0$ (b)  $\oint_S \mathbf{Q} \cdot d\mathbf{S} = 0$ (e)  $\nabla^2 \mathbf{Q} = 0$ (c)  $\nabla \times \mathbf{Q} = 0$ 

Answers: 3.1a-(vi), b-(vii), c-(v), d-(i), e-(ii), f-(iv), g-(iii), 3.2a-(vi), b-(v), c-(vii), d-(ii), e-(i), f-(iv), g-(iii), 3.3a-(v), b-(vi), c-(iv), d-(iii), e-(i), f-(ii), 3.4b, 3.5c, 3.6c, 3.7e, 3.8a, 3.9a, 3.10b.

## PROBLEMS

## Section 3.2—Differential Length, Area, and Volume

3.1 Calculate the areas of the following surfaces using the differential surface area  $dS$ :

(a)  $\rho = 2, 0 < z < 5, \pi/3 < \phi < \pi/2$ (b)  $z = 1, 1 < \rho < 3, 0 < \phi < \pi/4$ (c)  $r = 10, \pi/4 < \theta < 2\pi/3, 0 < \phi < 2\pi$ (d)  $0 < r < 4, 60^\circ < \theta < 90^\circ, \phi = \text{constant}$ 

3.2 Using the differential length  $dl$ , find the length of each of the following curves:

(a)  $\rho = 3, \pi/4 < \phi < \pi/2, z = \text{constant}$ (b)  $r = 1, \theta = 30^\circ, 0 < \phi < 60^\circ$ (c)  $r = 4, 30^\circ < \theta < 90^\circ, \phi = \text{constant}$

- 3.3 Calculate the area of the surface defined by  $\rho = 10$ ,  $\pi/4 < \phi < \pi/2$ ,  $0 < z < 2$ .
- 3.4 Find the length of a path from  $P_1(4, 0^\circ, 0)$  to  $P_2(4, 30^\circ, 0)$ .
- 3.5 Use the differential volume  $dv$  to determine the volumes of the following regions:
- $0 < x < 1$ ,  $1 < y < 2$ ,  $-3 < z < 3$
  - $2 < \rho < 5$ ,  $\pi/3 < \phi < \pi$ ,  $-1 < z < 4$
  - $1 < r < 3$ ,  $\pi/2 < \theta < 2\pi/3$ ,  $\pi/6 < \phi < \pi/2$

### Sections 3.3—Line, Surface, and Volume Integrals

- 3.6 Given that  $\mathbf{H} = x^2\mathbf{a}_x + y^2\mathbf{a}_y$ , evaluate  $\int_L \mathbf{H} \cdot d\mathbf{l}$ , where  $L$  is along the curve  $y = x^2$  from  $(0, 0)$  to  $(1, 1)$ .
- 3.7 If the integral  $\int_A^B \mathbf{F} \cdot d\mathbf{l}$  is regarded as the work done in moving a particle from  $A$  to  $B$ , find the work done by the force field

$$\mathbf{F} = 2xy\mathbf{a}_x + (x^2 - z^2)\mathbf{a}_y - 3xz^2\mathbf{a}_z$$

on a particle that travels from  $A(0, 0, 0)$  to  $B(2, 1, 3)$  along

- The segment  $(0, 0, 0) \rightarrow (0, 1, 0) \rightarrow (2, 1, 0) \rightarrow (2, 1, 3)$
  - The straight line  $(0, 0, 0)$  to  $(2, 1, 3)$
- 3.8 A vector field is represented by  $\mathbf{F} = \rho^2\mathbf{a}_\rho + z\mathbf{a}_\phi + \cos\phi\mathbf{a}_z$  newtons. Evaluate the work done or  $\int_L \mathbf{F} \cdot d\mathbf{l}$ , where  $L$  is from  $P(2, 0^\circ, 0)$  to  $Q(2, \pi/4, 3)$ . Assume that  $L$  consists of the arc  $\rho = 2$ ,  $0 < \phi < \pi/4$ ,  $z = 0$ , followed by the line  $\rho = 2$ ,  $\phi = \pi/4$ ,  $0 < z < 3$ .
- 3.9 If

$$\mathbf{H} = (x - y)\mathbf{a}_x + (x^2 + zy)\mathbf{a}_y + 5yza_z$$

evaluate  $\int_L \mathbf{H} \cdot d\mathbf{l}$  along the contour of Figure 3.28.

- 3.10 Determine the circulation of  $\mathbf{B} = xya_x - yza_y + xza_z$  around the path  $L$  on the  $x = 1$  plane, shown in Figure 3.29.
- 3.11 Let  $\mathbf{A} = y\mathbf{a}_x + z\mathbf{a}_y + x\mathbf{a}_z$ . Find the flux of  $\mathbf{A}$  through surface  $y = 1$ ,  $0 < x < 1$ ,  $0 < z < 2$ .

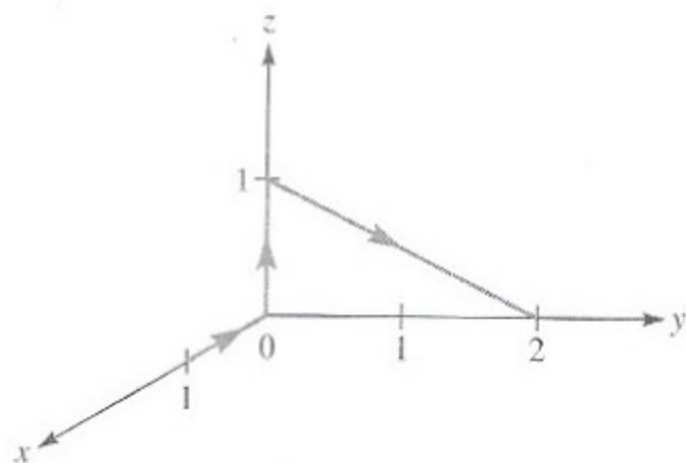


FIGURE 3.28 For Problem 3.9.

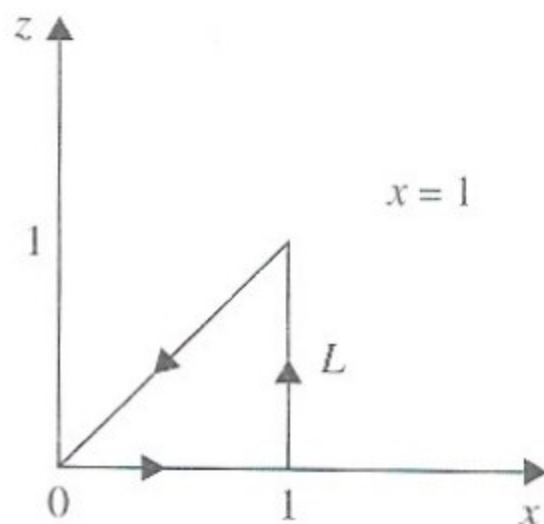


FIGURE 3.29 For Problem 3.10.



- 3.12 Find the flux of  $\mathbf{B} = 2xy\mathbf{a}_x + z^2y\mathbf{a}_y + 3x^2y\mathbf{a}_z$  over the surface defined by  $z = 1$ ,  $0 < x < 1$ ,  $0 < y < 2$ .
- 3.13 (a) Evaluate  $\int_V xy \, dv$ , where  $V$  is defined by  $0 < x < 1$ ,  $0 < y < 1$ ,  $0 < z < 2$ .  
 (b) Determine  $\int_V \rho z \, dv$ , where  $V$  is bounded by  $\rho = 1$ ,  $\rho = 3$ ,  $\phi = 0$ ,  $\phi = \pi$ ,  $z = 0$ , and  $z = 2$ .
- 3.14 Given that  $\mathbf{A} = r \sin \phi \mathbf{a}_r$ , find  $\int_V \mathbf{A} \, dv$ , where  $V$  is the spherical region  $r \leq 1$ . *Hint:* Convert  $\mathbf{a}_r$  to Cartesian coordinates.

### Section 3.5—Gradient of a Scalar

- 3.15 Calculate the gradient of:  
 (a)  $V_1 = 6xy - 2xz + z$   
 (b)  $V_2 = 10\rho \cos \phi - \rho z$   
 (c)  $V_3 = \frac{2}{r} \cos \phi$
- 3.16 Determine the gradients of the following fields:  
 (a)  $U = e^{x+2y} \cosh z$   
 (b)  $T = \frac{3z}{\rho} \cos \phi$   
 (c)  $W = \frac{5 \cos \phi}{r} + 2r^2 \sin \phi$
- 3.17 If  $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$  is the position vector of point  $(x, y, z)$ ,  $r = |\mathbf{r}|$ , and  $n$  is an integer, show that  $\nabla r^n = nr^{n-2}\mathbf{r}$ .
- 3.18 The temperature in an auditorium is given by  $T = x^2 + y^2 - z$ . A mosquito located at  $(1, 1, 2)$  in the auditorium desires to fly in such a direction that it will get warm as soon as possible. In what direction must it fly?
- 3.19 A family of planes is described by  $F = x - 2y + z$ . Find a unit normal  $\mathbf{a}_n$  to the planes.
- 3.20 Consider the scalar function  $T = r \sin \theta \cos \phi$ . Determine the magnitude and direction of the maximum rate of change of  $T$  at  $P(2, 6^\circ, 30^\circ)$ .
- 3.21 Let  $f = x^2y - 2xy^2 + z^3$ . Find the directional derivative of  $f$  at point  $(2, 4, -3)$  in the direction of  $\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$ .
- 3.22 (a) Using the gradient concept, prove that the angle between two planes  

$$ax + by + cz = d$$

$$\alpha x + \beta y + \gamma z = \delta$$
 is  

$$\theta = \cos^{-1} \frac{a\alpha + b\beta + c\gamma}{\sqrt{(a^2 + b^2 + c^2)(\alpha^2 + \beta^2 + \gamma^2)}}$$
 (b) Calculate the angle between two planes  $x + 2y + 3z = 5$  and  $x + y = 0$ .

## Section 3.6—Divergence of a Vector and Divergence Theorem

3.23 Evaluate the divergence of the following vector fields:

(a)  $\mathbf{A} = xya_x + y^2a_y + xza_z$

(b)  $\mathbf{B} = \rho z^2a_\rho + \rho \sin^2 \phi a_\phi + 2\rho z \sin^2 \phi a_z$

(c)  $\mathbf{C} = ra_r + r \cos^2 \theta a_\theta$

3.24 (a) If  $\mathbf{A} = x^2ya_x + xa_y + 2yza_z$ , find  $\nabla \cdot \mathbf{A}$  at point  $(-3, 4, 2)$ .

(b) Given that  $\mathbf{B} = 3\rho \sin \phi a_\rho - 5\rho^2 z a_\phi + 8z \cos^2 \phi a_z$ , find  $\nabla \cdot \mathbf{B}$  at point  $(5, 30^\circ, 1)$ .

(c) Let  $\mathbf{C} = r^2 \cos \phi a_r + 2ra_\phi$ , find  $\nabla \cdot \mathbf{C}$  at point  $(2, \pi/3, \pi/2)$ .

3.25 The heat flow vector  $\mathbf{H} = k\nabla T$ , where  $T$  is the temperature and  $k$  is the thermal conductivity. Show that if

$$T = 50 \sin \frac{\pi x}{2} \cosh \frac{\pi y}{2}$$

then  $\nabla \cdot \mathbf{H} = 0$ .

3.26 If  $\mathbf{A} = 2xa_x - z^2a_y + 3xya_z$ , find the flux of  $\mathbf{A}$  through a surface defined by  $\rho = 2$ ,  $0 < \phi < \pi/2$ ,  $0 < z < 1$ .

3.27 Let  $\mathbf{D} = 2\rho z^2a_\rho + \rho \cos^2 \phi a_z$ . Evaluate

(a)  $\oint_S \mathbf{D} \cdot d\mathbf{S}$

(b)  $\int_V \nabla \cdot \mathbf{D} dv$

over the region defined by  $2 \leq \rho \leq 5$ ,  $-1 \leq z \leq 1$ ,  $0 < \phi < 2\pi$ .

3.28 If  $\mathbf{H} = 10 \cos \theta a_r$ , evaluate  $\int_S \mathbf{H} \cdot d\mathbf{S}$  over a hemisphere defined by  $r = 1$ ,  $0 < \phi < 2\pi$ ,  $0 < \theta < \pi/2$ .

3.29 Evaluate both sides of the divergence theorem for the vector field

$$\mathbf{H} = 2xya_x + (x^2 + z^2)a_y + 2yza_z$$

and the rectangular region defined by  $0 < x < 1$ ,  $1 < y < 2$ ,  $-1 < z < 3$ .

3.30 Given that  $\mathbf{B} = \rho a_\rho + 10za_z$  evaluate both sides of the divergence theorem for the region defined by  $0 \leq \rho \leq 3$ ,  $0 \leq \phi \leq 2\pi$ ,  $0 \leq z \leq 4$ .

\*3.31 Apply the divergence theorem to evaluate  $\oint_S \mathbf{A} \cdot d\mathbf{S}$ , where  $\mathbf{A} = x^2a_x + y^2a_y + z^2a_z$  and  $S$  is the surface of the solid bounded by the cylinder  $\rho = 1$  and planes  $z = 2$  and  $z = 4$ .

3.32 Verify the divergence theorem for the function  $\mathbf{A} = r^2a_r + r \sin \theta \cos \phi a_\theta$  over the surface of a quarter of a hemisphere defined by  $0 < r < 3$ ,  $0 < \phi < \pi/2$ ,  $0 < \theta < \pi/2$ .

3.33 Calculate the total outward flux of vector

$$\mathbf{F} = \rho^2 \sin \phi a_\rho + z \cos \phi a_\phi + \rho za_z$$

through the hollow cylinder defined by  $2 \leq \rho \leq 3$ ,  $0 \leq z \leq 5$ .



## Section 3.7—Curl of a Vector and Stokes's Theorem

3.34 Evaluate the curl of the following vector fields:

- (a)  $\mathbf{A} = xya_x + y^2a_y + xza_z$   
 (b)  $\mathbf{B} = \rho z^2a_\rho + \rho \sin^2 \phi a_\phi + 2\rho z \sin^2 \phi a_z$   
 (c)  $\mathbf{C} = ra_r + r \cos^2 \theta a_\phi$

3.35 Evaluate  $\nabla \times \mathbf{A}$  and  $\nabla \cdot (\nabla \times \mathbf{A})$  if:

- (a)  $\mathbf{A} = x^2ya_x + y^2za_y - 2xza_z$   
 (b)  $\mathbf{B} = \rho^2za_\rho + \rho^3a_\phi + 3\rho z^2a_z$   
 (c)  $\mathbf{A} = \frac{\sin \phi}{r^2}a_r - \frac{\cos \phi}{r^2}a_\theta$

3.36 Let  $\mathbf{H} = \rho \sin \phi a_\rho + \rho \cos \phi a_\phi - \rho a_z$ ; find  $\nabla \times \mathbf{H}$  and  $\nabla \cdot \nabla \times \mathbf{H}$ .

3.37 Let  $\mathbf{A} = \frac{x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{(x^2 + y^2 + z^2)^{3/2}}$ ; show that  $\nabla \times \mathbf{A} = 0$ .

\*3.38 Given that  $\mathbf{F} = x^2ya_x - ya_y$ , find

- (a)  $\oint_L \mathbf{F} \cdot d\mathbf{l}$  where  $L$  is shown in Figure 3.30.  
 (b)  $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ , where  $S$  is the area bounded by  $L$ .  
 (c) Is Stokes's theorem satisfied?

3.39 Let  $\mathbf{A} = \rho \sin \phi a_\rho + \rho^2 a_\phi$ ; evaluate  $\oint_L \mathbf{A} \cdot d\mathbf{l}$  if  $L$  is the contour of Figure 3.31.

3.40 If  $\mathbf{F} = 2\rho za_\rho + 3z \sin \phi a_\phi - 4\rho \cos \phi a_z$ , verify Stokes's theorem for the open surface defined by  $z = 1$ ,  $0 < \rho < 2$ ,  $0 < \phi < 45^\circ$ .

3.41 Let  $\mathbf{A} = 4x^2e^{-y}a_x - 8xe^{-y}a_y$ . Determine  $\nabla \times [\nabla(\nabla \cdot \mathbf{A})]$ .

3.42 Let  $V = \frac{\sin \theta \cos \phi}{r}$ . Determine:

- (a)  $\nabla V$ , (b)  $\nabla \times \nabla V$ , (c)  $\nabla \cdot \nabla V$

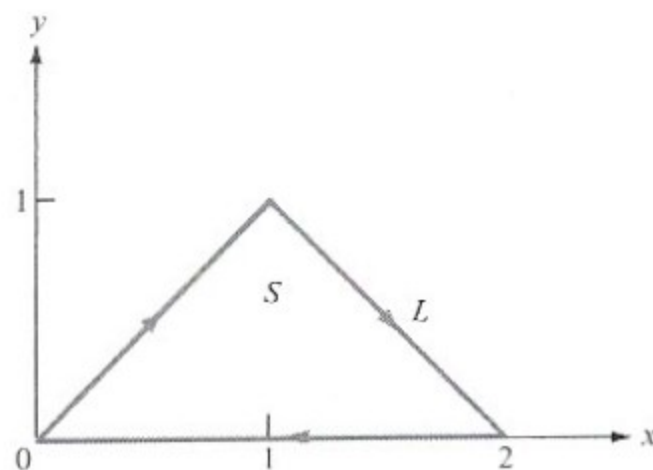


FIGURE 3.30 For Problem 3.38.

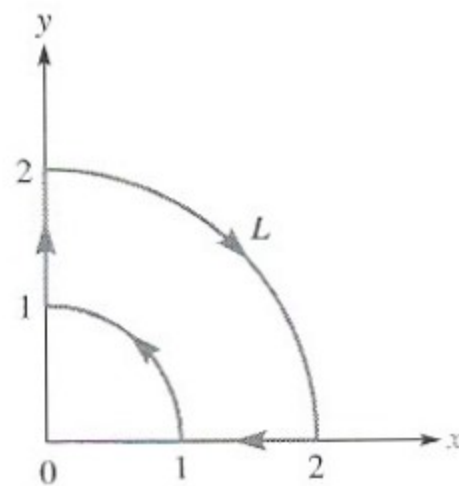


FIGURE 3.31 For Problem 3.39.



**\*\*3.43** A vector field is given by

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2}} [(x - y)\mathbf{a}_x + (x + y)\mathbf{a}_y]$$

Evaluate the following integrals:

- $\int_L \mathbf{Q} \cdot d\mathbf{l}$ , where  $L$  is the circular edge of the volume in the form of an ice cream cone shown in Figure 3.32.
- $\int_{S_1} (\nabla \times \mathbf{Q}) \cdot d\mathbf{S}$ , where  $S_1$  is the top surface of the volume
- $\int_{S_2} (\nabla \times \mathbf{Q}) \cdot d\mathbf{S}$ , where  $S_2$  is the slanting surface of the volume
- $\int_{S_1} \mathbf{Q} \cdot d\mathbf{S}$
- $\int_{S_2} \mathbf{Q} \cdot d\mathbf{S}$
- $\int_V \nabla \cdot \mathbf{Q} dv$

How do your results in parts (a) to (f) compare?

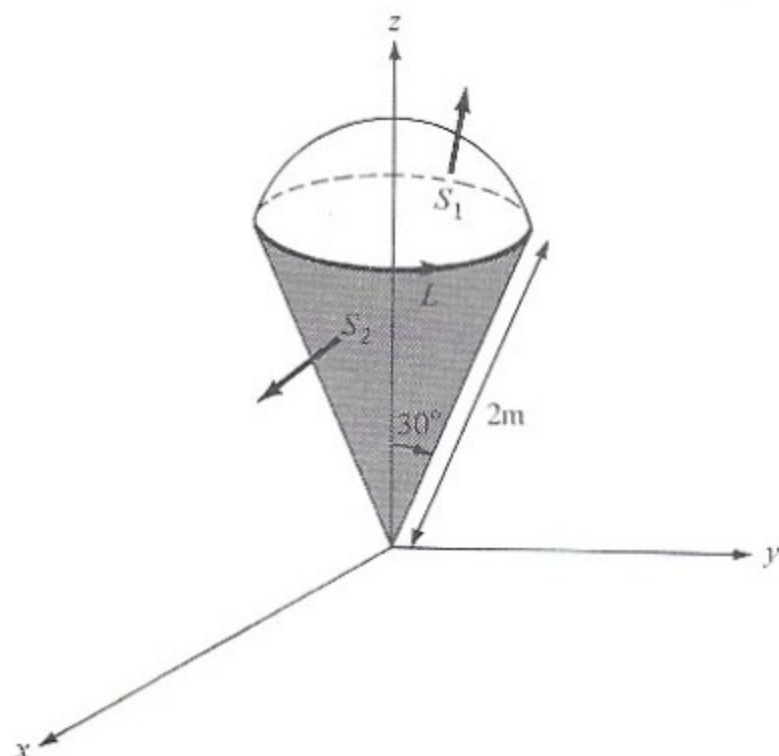
**\*3.44** A rigid body spins about a fixed axis through its center with angular velocity  $\omega$ . If  $\mathbf{u}$  is the velocity at any point in the body, show that  $\omega = 1/2 \nabla \times \mathbf{u}$ .

**3.45** Find the divergence and curl of

- $\mathbf{B} = \rho \cos \phi \mathbf{a}_\rho + \rho^2 \sin \phi \mathbf{a}_\phi - 4z \mathbf{a}_z$
- $\mathbf{F} = r^2 \sin \phi \mathbf{a}_r + 2r \cos \phi \mathbf{a}_\phi$

**3.46** For a vector field  $\mathbf{A}$  and a scalar field  $V$ , show in Cartesian coordinates that

- $\nabla \cdot (V \nabla V) = V \nabla^2 V + |\nabla V|^2$
- $\nabla \times (V \mathbf{A}) = V \nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$



**FIGURE 3.32** Volume in form of ice cream cone for Problem 3.43.

**\*\***Double asterisks indicate problems of highest difficulty.

3.47 If  $\mathbf{B} = x^2y\mathbf{a}_x + (2x^2 + y)\mathbf{a}_y - (y - z)\mathbf{a}_z$ , find

- (a)  $\nabla \cdot \mathbf{B}$
- (b)  $\nabla \times \mathbf{B}$
- (c)  $\nabla (\nabla \cdot \mathbf{B})$
- (d)  $\nabla \times \nabla \times \mathbf{B}$

### Section 3.8—Laplacian of a Scalar

3.48 Find  $\nabla^2 V$  for each of the following scalar fields:

- (a)  $V_1 = x^3 + y^3 + z^3$
- (b)  $V_2 = \rho z^2 \sin 2\phi$
- (c)  $V_3 = r^2 (1 + \cos \theta \sin \phi)$

3.49 Find the Laplacian of the following scalar fields and compute the value at the specified point.

- (a)  $U = x^3 y^2 e^{xz}$ ,  $(1, -1, 1)$
- (b)  $V = \rho^2 z (\cos \phi + \sin \phi)$ ,  $(5, \pi/6, -2)$
- (c)  $W = e^{-r} \sin \theta \cos \phi$ ,  $(1, \pi/3, \pi/6)$

3.50 If  $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$  is the position vector of point  $(x, y, z)$ ,  $r = |\mathbf{r}|$ , show that:

- (a)  $\nabla(\ln r) = \frac{\mathbf{r}}{r^2}$
- (b)  $\nabla^2(\ln r) = \frac{1}{r^2}$

3.51 Let  $V = xy^2z^3$ , evaluate  $\nabla V$  and  $\nabla^2 V$  at point  $P(1, 2, 3)$ .

3.52 Given that  $V = \rho^2 z \cos \phi$ , find  $\nabla V$  and  $\nabla^2 V$ .

3.53 If  $V = \frac{5 \cos \phi}{r^2}$ , find: (a)  $\nabla V$ , (b)  $\nabla \cdot \nabla V$ , (c)  $\nabla \times \nabla V$ .

3.54 Let  $U = 4xyz^2 + 10yz$ . Show that  $\nabla^2 U = \nabla \cdot \nabla U$ .

\*3.55 In cylindrical coordinates,

$$\nabla^2 \mathbf{A} = \left( \nabla^2 A_\rho - \frac{2}{\rho^2} \frac{\partial A_\phi}{\partial \phi} - \frac{A_\rho}{\rho^2} \right) \mathbf{a}_\rho + \left( \nabla^2 A_\phi + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \phi} - \frac{A_\phi}{\rho^2} \right) \mathbf{a}_\phi + \nabla^2 A_z \mathbf{a}_z$$

If  $\mathbf{G} = 2\rho \sin \phi \mathbf{a}_\rho + 4\rho \cos \phi \mathbf{a}_\phi + (z^2 + 1)\rho \mathbf{a}_z$ , find  $\nabla^2 \mathbf{G}$ .

3.56 According to eq. (3.64),  $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ . Show that

$\mathbf{A} = xz\mathbf{a}_x + z^2\mathbf{a}_y + yz\mathbf{a}_z$  satisfies this vector identity.



## Section 3.9—Classification of Vector Fields

3.57 Consider the following vector fields:

$$\mathbf{A} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$$

$$\mathbf{B} = 2\rho \cos \phi \mathbf{a}_\rho - 4\rho \sin \phi \mathbf{a}_\phi + 3\mathbf{a}_z$$

$$\mathbf{C} = \sin \theta \mathbf{a}_r + r \sin \theta \mathbf{a}_\phi$$

Which of these fields are (a) solenoidal, and (b) irrotational?

3.58 Given the vector field

$$\mathbf{G} = (16xy - z)\mathbf{a}_x + 8x^2 \mathbf{a}_y - x\mathbf{a}_z$$

(a) Is  $\mathbf{G}$  irrotational (or conservative)?

(b) Find the net flux of  $\mathbf{G}$  over the cube  $0 < x, y, z < 1$ .

(c) Determine the circulation of  $\mathbf{G}$  around the edge of the square  $z = 0, 0 < x, y < 1$ . Assume anticlockwise direction.

3.59 Show that the vector field  $\mathbf{F} = yz\mathbf{a}_x + xz\mathbf{a}_y + xy\mathbf{a}_z$  is both solenoidal and conservative.

3.60 A vector field is given by  $\mathbf{H} = \frac{10}{r^2} \mathbf{a}_r$ . Show that  $\oint_L \mathbf{H} \cdot d\mathbf{l} = 0$  for any closed path  $L$ .

3.61 Show that if  $\mathbf{A}$  and  $\mathbf{B}$  are irrotational, then  $\mathbf{A} \times \mathbf{B}$  is divergenceless or solenoidal.